assisted in doing so by agitations effected either by thermal or mechanical agency; hence—

Rest after suspension, aided by oscillations at intervals, diminishes the internal friction of a wire which has been recently suspended, or which after a long suspension has been subjected to considerable molecular agitation by either mechanical or thermal agency.

On the contrary, when a maximum molecular rotatory elasticity has been reached, molecular agitation, if carried beyond a certain limit, diminishes the elasticity; hence the results of "fatigue of elasticity;" and hence—

Mechanical shocks and rapid fluctuations of temperature beyond certain limits may considerably increase the internal friction, and, though to a much less extent, diminish the torsional elasticity.

The logarithmic decrement is independent of both the length and diameter of the wire.

II. "On Systems of Circles and Spheres." By R. LACHLAN, B.A., Fellow of Trinity College, Cambridge. Communicated by Professor A. CAYLEY, F.R.S. Received February 23, 1886.

(Abstract.)

This memoir is an attempt to develop the ideas contained in two papers to be found in the volume of Clifford's Mathematical Papers (Macmillan, 1882), viz., "On Power Coordinates" (pp. 546—555), and "On the Powers of Spheres" (pp. 332—336); the date of the former is stated to be 1866, and of the latter 1868, but the editor explains (see p. xxii, and note, p. 332) that though these papers probably contain the substance of a paper read to the London Mathematical Society, February 27, 1868, "On Circles and Spheres" ("Proc. L. M. S.," vol. ii, p. 61), they were probably not written out before 1876. It is possible, therefore, that Clifford may be indebted to Darboux for the conception of the "*power* of two circles," or spheres, as an extension of Steiner's use of the "*power* of a point with respect to a circle. Darboux was the first to give the definition of the power of two circles, in a paper "Sur les Relations entre les Groupes de Points, de Cercles, et de Sphères" ("Annales de l'École Normale Supérieure," vol. i, p. 323, 1872), in which some theorems analogous to the fundamental theorem of this memoir are proved.

This memoir is divided into three Parts: Part I consists of the discussion of systems of circles in one plane; Part II of systems of circles on the surface of a sphere; and Part III of systems of spheres.

The power of two circles is defined to be the square of the distance between their centres less the sum of the squares of their radii.

Denoting the power of two circles (1, 2) by $\pi_{1,2}$, it is proved that the power of any five circles (1, 2, 3, 4, 5) with respect to any other circles (6, 7, 8, 9, 10) are connected by the relation—

$$\begin{vmatrix} \pi_{1,6}, & \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10} \\ \pi_{2,6}, & \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10} \\ \pi_{3,6}, & \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10} \\ \pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10} \\ \pi_{5,6}, & \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10} \end{vmatrix} = 0$$

which may be conveniently written:—

$$\pi\binom{1,\ 2,\ 3,\ 4,\ 5}{6,\ 7,\ 8,\ 9,\ 10} = 0.$$

This is the fundamental theorem of the paper; it is shown that if the power of a straight line and a circle be defined as the perpendicular from the centre of the circle on the straight line, and the power of two straight lines as the cosine of the angle between them: then the theorem is true if any of the circles of either system be replaced by points, straight lines, or the line at infinity.

Several special systems of circles are then discussed, the most remarkable perhaps being the case when the circles (1, 2, 3, 4) being given, the circles (5, 6, 7, 8) are orthogonal to the former taken three at a time; then (x, y), denoting any other circles, the equation—

$$\pi\binom{x,\ 1,\ 2,\ 3,\ 4}{y,\ 5,\ 6,\ 7,\ 8} = 0$$

becomes $$\pi_{x,y} = \frac{\pi_{x,5}\cdot\pi_{y,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\cdot\pi_{y,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\cdot\pi_{y,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\cdot\pi_{y,4}}{\pi_{4,8}}$$

and as a particular case when the two circles (x, y) are replaced by the line at an infinity, we have

$$\frac{1}{\pi_{1,5}} + \frac{1}{\pi_{2,6}} + \frac{1}{\pi_{3,7}} + \frac{1}{\pi_{4,8}} = 0.$$

The general theorem is then applied to prove some properties of circles connected with three circles; a formula is given for the radius of a circle which passes through three of the points of intersection of three given circles; the eight circles which can be drawn to touch three circles are shown to be each touched by four of eight other circles, called Dr. Hart's circles, these arrange themselves in pairs; if ρ, ρ' be the radii of a pair of Dr. Hart's circles, and R, R' the radii of the corresponding pair of the eight circles passing through the points of intersection of the given circles, it is shown that

$$\frac{1}{\rho} - \frac{1}{\rho'} = 2\left(\frac{1}{R} - \frac{1}{R'}\right).$$

If (1, 2, 3, 4) denote any system of circles not having a common orthogonal circle, then defining the "power-coordinates" of a point as any multiples of its powers with respect to the system of reference (1, 2, 3, 4), it is deduced from the equation

$$\pi\binom{1,\ 2,\ 3,\ 4,\ 5}{1,\ 2,\ 3,\ 4,\ 6}=0,$$

that the coordinates of any point must satisfy a non-homogeneous linear relation, and a homogeneous quadric relation called the absolute. Also the equation of the first degree in power-coordinates represents a circle, unless it be satisfied by the coordinates of the line at infinity, and then it represents a straight line.

The equation of the second degree is shown to represent a bicircular quartic, or a circular cubic, some general properties are proved, and then the curves are classified. It is shown that the equation may be reduced to one of the forms

$$ax^2+by^2+cz^2+dw^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

the absolute being $\quad x^2+\ y^2+\ z^2+\ w^2=0$;

or $\quad ax^2+by^2+cz^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$

$$ax^2+2fyz \quad =0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(C)}$$

the absolute being $\quad x^2+y^2-4zw=0.$

The different curves are then discussed in detail, there being nine species in all, three in each group (A), (B), or (C).

Part II contains merely the extension of the results of Part I to spherical geometry; the power of two circles on a sphere is defined to be the product of $\tan r$, $\tan r'$, $\cos\omega$, where r, r', are the radii, ω their angle of intersection; the power of a small circle radius r, and a great circle is, however, defined as $\tan r\cos\omega$; and of two great circles as $\cos\omega$.

The fundamental theorem is as before

$$\pi\binom{1,\ 2,\ 3,\ 4,\ 5}{6,\ 7,\ 8,\ 9,\ 10}=0,$$

connecting the powers of two systems of circles.

Consequently the results obtained previously are extended with but slight modification.

In Part III the method of Part I is applied to spheres; it is proved at once that the powers of any systems of spheres must satisfy the relation

$$\pi\binom{1,\ 2,\ 3,\ 4,\ 5,\ 6}{7,\ 8,\ 9,\ 10,\ 11,\ 12}=0,$$

and any of the spheres may be replaced by planes, or the plane at infinity.

Several results obtained in Part I are easily extended, with one

exception; there are eight pairs of spheres which touch four given spheres, but except in very special cases no spheres exist analogous to Dr. Hart's circles.

The discussion of the equation of the first degree in power-coordinates is much the same as that in Part I. The reduction, however, of the general equation of the second degree is more complicated; there are four distinct forms to which the equation may be reduced.

$$ax^2+by^2+cz^2+dw^2+ev^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

the equation of the absolute being

$$x^2+y^2+z^2+w^2+v^2=0;$$

this is the general cyclide, of either the fourth or third order; if $d=e$, it has two cnic-nodes. and if $b=c$, $d=e$, it has four cnic-nodes; but in this case the sphere $x=0$ must be imaginary.

$$ax^2+by^2+cz^2+dw^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\beta)$$

the equation of the absolute being

$$x^2+y^2+z^2-4wv=0.$$

This is the general case of a cyclide having one cnic-node, if $b=c$ it has three nodes; the former case is the inverse of a central quadric, the latter the inverse of a central quadric of revolution: the spheres x, y, z are real in this case.

$$ax^2+by^2+2hzw=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\gamma)$$

the equation of the absolute being

$$x^2+y^2+z^2-4wv=0.$$

This represents a cyclide having two principal spheres and a binode; if a or $b=0$ the node is a unode.

$$ax^2+2hyz+dw^2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\delta)$$

the equation of the absolute being

$$x^2+y^2+z^2-4wv=0.$$

This represents a cyclide having only one principal sphere; and a cnic-node, which becomes a binode when $a=0$, and a unode when $h=0$.

The different species of cyclides are then briefly discussed in detail.